Mathematics 222B Lecture 8 Notes

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1 Hardy's Inequality and Introduction to Elliptic PDEs

1.1 Hardy's inequality

Last time, we introduced Hardy's inequality.

Theorem 1.1 (Hardy's inequality)**.** *Let $u \in C_c^\infty(\mathbb{R}^d)$ with $d > 2$. Then*

$$\left\| \frac{1}{|x|} u \right\|_{L^2} \leq C \|Du\|_{L^2}.$$

This is a sharp inequality. We will see what the extremizer looks like.

Proof. Switch to polar coordinates (r, ω). It suffices to show that this inequality holds with the radial derivative: For each fixed ω,

$$\int \frac{1}{r^2} u^2 r^{d-1}\, dt \leq C \int |\partial_r u|^2 r^{d-1}\, dr,$$

and then we integrate over ω on both sides. The idea is to complete the square. We will subtract one side from the other and show it is ≥ 0. Without motivation, let's examine

$$(\partial_r u + \tfrac{\alpha}{r} u)^2 = (\partial_r u)^2 + \frac{2\alpha}{r} u\, \partial_r u + \frac{\alpha^2}{r^2} u^2.$$

The left hand side is ≥ 0. Now integrate both sides:

$$\begin{aligned} 0 &\leq \int (\partial_r u + \tfrac{\alpha}{r} u)^2 r^{d-1}\, dr \\ &= \int \left((\partial_r u)^2 + \frac{2\alpha}{r} \underbrace{u \partial_r u}_{\frac{1}{2}\partial_r u^2} + \frac{\alpha^2}{r^2} u^2 \right) r^{d-1}\, dr \\ &= \int (\partial_r u)^2 r^{d-1}\ dr + \alpha^2 \int \frac{1}{r^2} u^2 r^{d-1}\, dt + \alpha \int_0^\infty \partial_r u^2 r^{d-2}\, dr \end{aligned}$$

We want to integrate by parts. Since $d > 0$, the boundary term will be 0. In particular, $\int_0^\infty \partial_r u^2 r^{d-2}\, dr = \cancel{u^2 r^{d-2}|_0^\infty} - (d-2)\int_0^\infty u^2 r^{d-3}\, dr$.

$$= \int (\partial_r u)^2 r^{d-1}\, dr - ((d-2)\alpha - \alpha^2) \int_0^\infty \frac{1}{r^2} u^2 r^{d-1}\, dr$$

Really, what we need here is $(d-2) > 0$ because we want the coefficient of α in the above quadratic term to be positive. We can upper bound this by plugging in $\alpha = \frac{d-2}{2}$. We can also upper bound $\int_0^\infty \frac{1}{r^2} u^2 r^{d-1}\, dr \le (\frac{2}{d-2})^2 \int (\partial_r u)^2 r^{d-1}\, dr$. $\square$

Remark 1.1. Not only do we get the inequality, but we also get that

$$\left(\frac{d-2}{2}\right)^2 \int_0^\infty \frac{1}{r^2} u^2\, dr = \int_0^\infty (\partial_r u)^2 r^{d-1}\, dr - \int_0^\infty \left(\partial_r u + \frac{d-2}{2r} u\right)^2 r^{d-1}\, dr.$$

This tells us that the extremizer is $r^{-(d-2)/2}$. However, this is not an element of H^1, so we can get near extremizers by including appropriate cutoffs.

1.2 Linear elliptic equations

Elliptic PDEs are a generalization of the Laplace equation $-\Delta u = f$.

Definition 1.1. The **symbol** of a partial differential operator is what we get when we replace ∂_j with $i\xi_j$.

It turns out that an important property is that $-\Delta \sum_j \partial_j \partial_j$ has (principal) symbol $-\sum_j (i\xi_j)^2 = |\xi|^2$. What's important is that $|\xi|^2$ is nonzero and thus invertible for $\xi \neq 0$:

$$|\xi|^2 \widehat{u} = \widehat{f} \implies \widehat{u} = \frac{1}{|\xi|^2} \widehat{f}.$$

This leads to the general definition of ellipticity of a partial differential operator.

Suppose that P is a linear partial differential operator such that if $u = (u^I)_{I=1}^N : U \to \mathbb{R}^N$, then (Pu) takes values in $\mathbb{R}^N$ and

$$(Pu)^I = \underbrace{\sum_{\substack{J,\alpha \\ |\alpha|=K}} A^I_{J,\alpha_1,\dots,\alpha_d} \partial^\alpha u^J}_{\text{principal part}} + (\text{lower order terms}).$$

Here, K is called the **order** of P.

Definition 1.2. The **principal symbol** of an operator is

$$\sigma_{\text{prin}}(P) = i^K \sum_{\substack{\alpha \\ \alpha=K}} A^I_{J,\alpha_1,\dots,\alpha_d}(x) \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d}.$$

Here, we allow the coefficients to be functions of x. We say that P is **elliptic** if $\sigma_{\text{prin}}(P)$ is invertible for all $x \in U$ and $\xi \neq 0$.

The case $N = 1$ is called the **scalar case**, where this looks like

$$Pu = \sum_{|\alpha|=K} a_\alpha(x)\partial^\alpha u.$$

Then the principal symbol is

$$\sigma_{\text{prin}}(P) = i^K \sum_\alpha a_\alpha(x)\xi^\alpha$$

The first nontrivial example is when $K = 2$, so

$$Pu = a^{i,j}\partial_i\partial_j + b^i\partial_j + c.$$

In this case, ellipticity is equivalent to $a^{i,j}\xi_i\xi_j \neq 0$ for all $x \in U$ and $\xi \neq 0$. This is equivalent to $a = [a^{i,j}]$ being a positive definite matrix for all $x \in U$.

We will assume that a is a symmetric matrix and require the following property.

Definition 1.3. Uniform ellipticity, is the property that there exists a uniform constant $\lambda > 0$ such that $a^{i,j}\xi_i\xi_j \geq \lambda$ for all $x \in U$ and $|\xi| = 1$.

This is equivalent to saying that all eigenvalues of the matrix $a(x)$ are bounded below by λ.

Why do we care about elliptic PDEs?

1. These arise naturally in optimization problems in math, physics, etc. In the latter part of the course, we will discuss these in the context of calculus of variations.

2. They also often arise as a part of evolutionary problems.

Example 1.1 (Incompressible Euler equations)**.** Let $u : \mathbb{R}_t \times \mathbb{R}^3 \to \mathbb{R}^3$ represent the velocity of a fluid element at each point in time and space. This follows the equation

$$\begin{cases} \partial_t + u \cdot \nabla u + \nabla p = 0 \\ \nabla \cdot u = 0 \end{cases}$$

This is one of the most infamous PDEs because of how difficult it is to understand.

How do we figure out p? Take the divergence of the first equation to get that

$$-\Delta p = \nabla(u \cdot \nabla u).$$

This is the **pressure equation**.

We will cover:

- Boundary value problems for elliptic PDEs, existence, and uniqueness.
- Regularity properties of solutions to elliptic PDEs. If $Pu = f$, where P is elliptic of order K, then we will have **elliptic regularity**[1]: If f has regularity of order k (so

[1] Elliptic regularity holds even for systems.

$f \in H^k$), then u has regulariity $k + K$.

- Maximum principles (mostly for the scalar case, $N = 1$).

If we have time, we will discuss topics such as

- Unique continuation.
- Spectral theory.

We will mostly follow Evans' textbook, but we will deviate sometimes on a few topics.

1.3 Boundary value problems and a priori estimates for elliptic PDEs

Assume $d \geq 2$ and $N = 1$ (scalar case). Also assume uniform ellipticity of P and some "nice" regularity for the coefficients a, b, c. We will focus mostly on the case where U is a bounded domain in $\mathbb{R}^d$ with "nice" boundary.

When it comes to boundary value problems, you cannot prescribe both function values and values of the normal derivative at the boundary; this stems from the various uniqueness properties that arise for these PDEs. We will mostly focus on **Dirichlet boundary problems**,

$$\begin{cases} Pu = f & \text{in } U \\ u = g & \text{on } \partial U. \end{cases}$$

We will focus less on boundary problems such as **Neumann boundary problems**,

$$\begin{cases} Pu = f & \text{in } U \\ \frac{\partial}{\partial \nu} u = g & \text{on } \partial U. \end{cases}$$

We will study solvability for $u \in H^1(U)$. We will first study the Dirichlet boundary value problem ($u|_{\partial u} = g$ is okay due to the trace theorem). We will later discuss the Neumann boundary value problem, which needs to be studied in H^2 because we need to use the trace theorem on the derivative.

The standard reduction is that it suffices to understand $g = 0$. This is because if we take any extension (with correct regularity) $\widetilde{g} : \overline{U} \to \mathbb{R}$ of g, then we can work with $v = u0\widetilde{g}$ and solve the problem

$$\begin{cases} Pv = f + P\widetilde{g} = \widetilde{f} & \text{in } U \\ v = 0 & \text{on } \partial U. \end{cases}$$

Definition 1.4. P is in **divergence form** if

$$Pu = \partial_i(a^{i,j}\partial_j u) + \partial_i(b^i u) + c.$$

Note that if a is smooth, then

$$a = a^{i,j}\partial_i\partial_j + (\partial_j a^{i,j} + b^i)\partial_i u + (\partial_i b^i + c)u.$$

Our discussion of existence and uniqueness of the Dirichlet boundary value problem would be based on a-priori estimates.

Theorem 1.2 (a-priori estimate)**.** *Suppose that $u \in H^1$ solves the Dirichlet boundary problem, and assume that $b, c \in L^\infty$ with $\|b\|_{L^\infty} + \|c\|_{L^\infty} \leq A$. Then there exist constants $C > 0$ and $\gamma \geq 0$ such that*

$$\|u\|_{H^1(U)} \leq C\|f\|_{H^{-1}} + \gamma\|u\|_{L^2(U)}.$$

Proof. The proof is essentially integration by parts. We can use approximation to justify the integration by parts. Write

$$\begin{aligned}\int_U Pu\,dx &= \int_U (\partial_j(a^{i,j}\partial_i u + b^j u) + cu)u\,dx \\ &= \int -a^{i,j}\partial_i u\partial_j u - b^j u\partial_j u + cuu\,dx\end{aligned}$$

Uniform ellipticity tells us that $\lambda|Du|^2 \leq a^{i,j}\partial_i u\partial_j u$; integrate this to take care of the first term. The second term can be dealt with using Cauchy-Schwarz, and the third term is $\gamma\|u\|_{L^2}^2$.

Putting this all together gives

$$\begin{aligned}\lambda\|Du\|_{L^2(U)}^2 &\leq C\|f\|_{H^{-1}}\|u\|_{H^1} + \int_U |b||\partial u||u|\,dx + \int_U |c||u|^2\,dx \\ &\leq C\|f\|_{H^{-1}}\|u\|_{H^1} + A\underbrace{\int_U |\partial u||u|\,dx}_{\leq\|\partial u\|\|u\|_{L^2}} + A\underbrace{\int_U |u|^2\,dx}_{\leq\gamma\|u\|_{L^2}^2}.\end{aligned}$$

If we make γ large enough so that we have put an $\|u\|_{L^2}^2$ on the right hand side and abosrb the second term, we get

$$\|u\|_{H^1(U)}^2 \leq C\|f\|_{H^{-1}}\|u\|_{H^1} + \gamma\|u\|_{L^2}\|u\|_{H^1}. \qquad \square$$

Remark 1.2. We can alter this argument to only require $b \in L^{d+}$ and $c \in L^{d/2+}$.